

On a class of mixed Choquard-Schrödinger-Poisson systems

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Abstract

We study the system

$$\begin{cases} -\Delta u + u + K(x)\phi|u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ -\Delta\phi = K(x)|u|^q & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $\alpha \in (0, N)$, $p, q > 1$ and $K \geq 0$. Using a Pohozaev type identity we first derive conditions in terms of p, q, N, α and K for which no solutions exist. Next, we discuss the existence of a ground state solution by using a variational approach.

Keywords: Choquard equation; Schrödinger-Poisson equation; Pohozaev identity; ground state solution

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1 Introduction

In this paper we are concerned with the following system

$$\begin{cases} -\Delta u + u + K(x)\phi|u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ -\Delta\phi = K(x)|u|^q & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $p, q > 1$ are real numbers and $K \geq 0$ satisfies some more properties as we shall state below. Here $I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R}$ is the *Riesz potential* of order $\alpha \in (0, N)$, $N \geq 3$, given by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{with } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}. \quad (1.2)$$

When $K \equiv 0$, system (1.1) reduces to the single equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \quad (1.3)$$

which bears the name *Choquard* or *Choquard-Pekar equation*.

For $N = 3$, $p = \alpha = 2$, equation (1.3) was introduced in 1954 by S.I. Pekar [28] as a model in quantum theory of a Polaron at rest (see also [12]). In 1976, P. Choquard used

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(1.3) in a certain approximation to Hartree-Fock theory of one component plasma (see [16]). In 1996, equation (1.3) appears in a different context, being employed by R. Penrose [29] as a model of self-gravitating matter (see, e.g., [14, 22]) and it is known in this context as the *Schrödinger-Newton equation*.

If u solves (1.3), then the function ψ defined by $\psi(t, x) = e^{it}u(x)$ is a solitary wave of the focussing time dependent Hartree-Fock equation

$$i\psi_t + \Delta\psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N.$$

The Choquard equation (1.3) has been investigated for a few decades by variational methods starting with the pioneering works of E.H. Lieb [16] and P.-L. Lions [17, 18]. More recently, new and improved techniques have been devised to deal with various forms of (1.3) (see, e.g., [1, 23, 25, 26, 27, 31] and the references therein). In [23] existence, regularity, positivity, asymptotic behavior and radial symmetry of solutions to (1.1) is discussed for optimal range of parameters. We also mention here the works [10, 11] where the fractional version of (1.3) is considered. For a nonvariational approach to Choquard equation the reader may consult [13, 19, 24].

Back to (1.1), we should point out that since for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $I_\alpha * \varphi \rightarrow \varphi$ as $\alpha \rightarrow 0$, the system

$$\begin{cases} -\Delta u + u + K(x)\phi|u|^{q-2}u = |u|^{2p-2}u & \text{in } \mathbb{R}^N, \\ -\Delta\phi = K(x)|u|^q & \text{in } \mathbb{R}^N, \end{cases} \quad (1.4)$$

may be seen as a formal limit of (1.1) when $\alpha \rightarrow 0$. The nonlocal nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + V_{ext}(x)\psi + (I_2 * |\psi|^2)\psi - |\psi|^{p-2}\psi = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

is used as an approximation to Hartree-Fock model of a quantum many-body system of electrons under the presence of an external potential V_{ext} (see [15]). In such a setting, (1.4) and its stationary counterpart bear the name of Schrödinger-Poisson-Slater [5], Schrödinger-Poisson- X_α [3, 20], or Maxwell-Schrödinger-Poisson [2, 7] equations. The convolution term in (1.4) represents the Coulombic repulsion between the electrons. The local term $|u|^{2p-2}u$ was introduced by Slater [30] as a local approximation of the exchange potential in the Hartree-Fock model [5, 20].

Notations. Throughout in this paper we use the following notations.

- $H^1(\mathbb{R}^N)$ denotes the standard Sobolev space endowed with the usual norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx.$$

We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\mathbb{R}^N)$ and its dual $H^{-1}(\mathbb{R}^N)$.

- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the Hilbert space

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$$

endowed with the standard norm

$$\|u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

and the associated scalar product

$$(u, v)_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v.$$

- $L^s(\mathbb{R}^N)$ is the usual Lebesgue space in \mathbb{R}^N of order $s \in [1, \infty]$ whose norm will be denoted by $\|\cdot\|_s$.

2 Main Results

Our first result provides sufficient conditions for the nonexistence of solutions to (1.1).

Theorem 2.1. *Assume $K \in C^1(\mathbb{R}^N)$, $K \geq 0$. If one of the following holds*

- (i) $x \cdot \nabla K(x) + \gamma K(x) \geq 0$ in \mathbb{R}^N for some $\gamma \in (-\infty, \frac{N+2}{2})$ and

$$p \geq \frac{N+\alpha}{N-2} \quad \text{and} \quad \frac{N+\alpha}{p} \leq \frac{N+2-2\gamma}{q}; \quad (2.1)$$

- (ii) $x \cdot \nabla K(x) + \gamma K(x) \leq 0$ in \mathbb{R}^N for some $\gamma \in \mathbb{R}$ and

$$p \leq \frac{N+\alpha}{N} \quad \text{and} \quad \frac{N+\alpha}{p} \geq \frac{N+2-2\gamma}{q}; \quad (2.2)$$

then, the only solution (u, ϕ) of (1.1) that satisfies

$$u \in H^1(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N), \quad \phi \in H^1(\mathbb{R}^N) \quad (2.3)$$

and

$$K(x)|u|^q \in L^{\frac{2N}{N+2}}(\mathbb{R}^N), \quad |\nabla u| \in H_{loc}^1(\mathbb{R}^N) \cap L_{loc}^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N) \quad (2.4)$$

is $u \equiv \phi \equiv 0$.

By taking $K \equiv 0$, for suitable choice of γ in (2.1) and (2.2) we obtain that if $p \geq \frac{N+\alpha}{N-2}$ or $p \leq \frac{N+\alpha}{N}$ then the only solution of (1.3) which satisfies $u \in H^1(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ and $|\nabla u| \in H_{loc}^1(\mathbb{R}^N) \cap L_{loc}^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ is the trivial one. We thus recover the result in [23, Theorem 2].

By taking $\gamma = 0$ in Theorem 2.1 we obtain:

Corollary 1. *Let $K \equiv \text{const} > 0$. If one of the following conditions hold*

$$p \geq \frac{N + \alpha}{N - 2} \quad \text{and} \quad \frac{N + \alpha}{p} \leq \frac{N + 2}{q};$$

or

$$p \leq \frac{N + \alpha}{N} \quad \text{and} \quad \frac{N + \alpha}{p} > \frac{N + 2}{q},$$

then the only solution (u, ϕ) of (1.1) satisfying (2.3)-(2.4) is the trivial one.

Corollary 2. *Let $K(x) = (1 + |x|^2)^{-\gamma/2}$. If $\gamma \in [0, \frac{N+2}{2})$ and (2.1) holds or $\gamma \leq 0$ and (2.2) holds then the only solution (u, ϕ) of (1.1) satisfying (2.3)-(2.4) is the trivial one.*

Let us now discuss the existence of a solution to (1.1). Crucial to our approach will be the Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^N} |I_\alpha * u|^{\frac{Ns}{N-\alpha s}} \leq C \left(\int_{\mathbb{R}^N} |u|^s \right)^{\frac{N}{N-\alpha s}} \quad \text{for any } u \in L^s(\mathbb{R}^N), s \in (1, \frac{N}{s}) \quad (2.5)$$

which also implies

$$\left| \int_{\mathbb{R}^N} (I_\alpha * u)v \right| \leq C \|u\|_s \|v\|_t \quad \text{for any } u \in L^s(\mathbb{R}^N), v \in L^t(\mathbb{R}^N), \frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}. \quad (2.6)$$

It is more convenient to reduce our system (1.1) to a single equation. More exactly, for any $u \in H^1(\mathbb{R}^N)$ define

$$T_u : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}, \quad T_u(v) = \int_{\mathbb{R}^N} K(x) |u|^q v dx.$$

If $K \in L^r(\mathbb{R}^N)$, with

$$\frac{1}{r} + \frac{q+1}{2^*} = 1 \quad \text{and} \quad 1 < q < \frac{N+2}{N-2}, \quad (2.7)$$

then, by Hölder and Sobolev inequality one gets that T_u is linear and continuous. By Lax-Milgram theorem, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$T_u(v) = (\phi_u, v)_{\mathcal{D}^{1,2}} \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N). \quad (2.8)$$

As a result, ϕ_u solves

$$-\Delta \phi_u = K(x) |u|^q \quad \text{in } \mathbb{R}^N,$$

and

$$\phi_u(x) = A_2 \int_{\mathbb{R}^N} \frac{K(y) |u|^p(y)}{|x-y|^{N-2}} dy \quad \text{where } A_2 \text{ corresponds to (1.2).}$$

Hence

$$\phi_u = I_2 * (K |u|^q). \quad (2.9)$$

More properties of ϕ_u are given in Lemma 3.1 below. We should finally note that with ϕ_u given by (2.9), system (1.1) reduces implicitly to the single equation

$$-\Delta u + u + K(x)\phi_u|u|^{q-2}u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \quad (2.10)$$

Let us remark that (2.10) has a variational structure. If $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and q, r satisfy (2.7) then functional

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) + \frac{1}{2q} \int_{\mathbb{R}^N} K(x)\phi_u|u|^q - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

is well defined for all $u \in H^1(\mathbb{R}^N)$ and any critical point u of \mathcal{J} is a weak solution to (2.10).

Our existence result is the following.

Theorem 2.2. *Assume $1 < q < \frac{N+2}{N-2}$, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, $q < p$ and $K \in L^r(\mathbb{R}^N)$, with r given by (2.7). Then, there exists $M > 0$ such that for any $\|K\|_r < M$ problem (1.1) has a solution $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, u is a ground state of (2.10).*

In order to deal with the lack of compactness of $H^1(\mathbb{R}^N)$ into the Lebesgue spaces $L^s(\mathbb{R}^N)$, $2 \leq s \leq 2^*$, we rely on a careful analysis of the Palais-Smale (in short (PS)) sequences for \mathcal{J} restricted to its Nehari manifold \mathcal{N} . Roughly speaking, we have that any (PS) sequence of $\mathcal{J}|_{\mathcal{N}}$ either converges strongly to its weak limit or differs from it by a finite number of sequences, which are nothing but translated solutions of (1.3), centered at points whose distances from the origin and whose interdistances go to infinity (see Proposition 5.2). Then, a further evaluation of the energy levels of \mathcal{J} allows us to locate some ranges for which the compactness is still preserved. Such an approach was successfully applied for the Schrödinger-Poisson system (1.4) in [8, 9] and recently adapted to the study of the non-autonomous fractional Choquard equation in [10]. Unlike the approach in [10] where a direct energy estimation is possible due to the presence of suitable non-autonomous terms, we shall rely essentially on several nonlocal Brezis-Lieb type results as we describe in Section 3.2.

The remaining part of the paper is organised as follows. Section 3 contains some preliminary results which we will use in the study of the existence of a ground state to (1.1). Sections 4 and 5 contain the proofs of our main results.

3 Preliminary results

3.1 Some properties of ϕ_u

Lemma 3.1. *We have*

- (i) $\phi_u \geq 0$ for any $u \in H^1(\mathbb{R}^N)$;
- (ii) $\phi_{tu} = t^q \phi_u$ for any $t > 0$;
- (iii) if $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $\phi_{u_n} \rightarrow \phi_u$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. (i) and (ii) follow from the definition of ϕ_u .

(iii) For a proof of this part in dimension $N = 3$ the reader may consult [8, Proposition 2.2(a)]. Here we provide a different argument.

Let us note first that from the definition of ϕ_u in (2.8) we deduce

$$\|\phi_u\|_{\mathcal{D}^{1,2}} = \|T_u\|_{\mathcal{L}(\mathcal{D}^{1,2})}.$$

For any $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ we have

$$\begin{aligned} |T_{u_n}(v) - T_u(v)| &\leq \int_{\mathbb{R}^N} K(x) ||u_n|^q - |u|^q| |v| \\ &\leq \|v\|_{\mathcal{D}^{1,2}} \left(\int_{\mathbb{R}^N} K(x)^{\frac{2N}{N+2}} ||u_n|^q - |u|^q|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}}. \end{aligned}$$

Using the continuous embedding of $H^1(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$, $2 \leq s \leq 2^*$ and Lemma 3.3 below, it follows that

$$||u_n|^q - |u|^q|^{\frac{2N}{N+2}} \rightharpoonup 0 \quad \text{weakly in } L^{\frac{N+2}{q(N-2)}}(\mathbb{R}^N).$$

Thus, since $K^{\frac{2N}{N+2}} \in L^{\frac{r(N+2)}{2N}}(\mathbb{R}^N)$ we deduce

$$\begin{aligned} \|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}} &= \|T_{u_n} - T_u\|_{\mathcal{L}(\mathcal{D}^{1,2})} \\ &\leq \left(\int_{\mathbb{R}^N} K(x)^{\frac{2N}{N+2}} ||u_n|^q - |u|^q|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \rightarrow 0. \end{aligned}$$

□

3.2 Some nonlocal versions of Brezis-Lieb lemma

In this part we collect some useful results in dealing with the existence of a ground state solution to (1.3).

We first recall the concentration-compactness lemma of P.-L. Lions formulated in an inequality setting.

Lemma 3.2. ([18, Lemma 1.1], [23, Lemma 2.3])

Let $s \in [2, 2^*]$. There exists a constant $C > 0$ such that for any $u \in H^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} |u|^s \leq C \|u\| \left(\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u|^s \right)^{1 - \frac{2}{s}}.$$

Lemma 3.3. ([4, Proposition 4.7.12])

Let $s \in (1, \infty)$. Assume (w_n) is a bounded sequence in $L^s(\mathbb{R}^N)$ that converges to w almost everywhere. Then $w_n \rightharpoonup w$ weakly in $L^s(\mathbb{R}^N)$.

Using a similar proof to that in the original Brezis-Lieb lemma [6, Theorem 2] (see also [32, Proposition 4.7.30]) we have

Lemma 3.4. (Local Brezis-Lieb lemma)

Let $s \in (1, \infty)$. Assume (w_n) is a bounded sequence in $L^s(\mathbb{R}^N)$ that converges to w almost everywhere. Then, for every $q \in [1, s]$ we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} ||w_n|^q - |w_n - w|^q - |w|^q|^\frac{s}{q} = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} ||w_n|^{q-1}w_n - |w_n - w|^{q-1}(w_n - w) - |w|^{q-1}w|^\frac{s}{q} = 0.$$

A first nonlocal version of Brezis-Lieb lemma in the literature appeared in [23] (see also [21]) and reads as follows.

Lemma 3.5. (Nonlocal Brezis-Lieb lemma, [23, Lemma 2.4])

Let $\alpha \in (0, N)$ and $p \in [1, \frac{2N}{N+\alpha})$. Assume (u_n) is a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ that converges almost everywhere to some $u : \mathbb{R}^N \rightarrow \mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |(I_\alpha * |u_n|^p)|u_n|^p - (I_\alpha * |u_n - u|^p)|u_n - u|^p - (I_\alpha * |u|^p)|u|^p| = 0.$$

Below we state and prove another nonlocal version of Brezis-Lieb lemma.

Lemma 3.6. Let $\alpha \in (0, N)$ and $p \in [1, \frac{2N}{N+\alpha})$. Assume (u_n) is a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ that converges almost everywhere to u . Then, for any $h \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_nh = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}uh.$$

Proof. Using $h = h^+ - h^-$, it is enough to prove our lemma for $h \geq 0$. Denote $v_n = u_n - u$ and observe that

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_nh &= \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)](|u_n|^{p-2}u_nh - |v_n|^{p-2}v_nh) \\ &\quad + \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)]|v_n|^{p-2}v_nh \\ &\quad + \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^{p-2}u_nh - |v_n|^{p-2}v_nh)]|v_n|^p \\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^{p-2}v_nh. \end{aligned} \tag{3.1}$$

Apply Lemma 3.4 with $q = p$, $s = \frac{2Np}{N+\alpha}$ by taking respectively $(w_n, w) = (u_n, u)$ and then $(w_n, w) = (u_nh^{1/p}, uh^{1/p})$. We find

$$\begin{cases} |u_n|^p - |v_n|^p \rightarrow |u|^p \\ |u_n|^{p-2}u_nh - |v_n|^{p-2}v_nh \rightarrow |u|^{p-2}uh \end{cases} \quad \text{strongly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Using now the Hardy-Littlewood-Sobolev inequality (2.5) we obtain

$$\begin{cases} I_\alpha * (|u_n|^p - |v_n|^p) \rightarrow I_\alpha * |u|^p \\ I_\alpha * (|u_n|^{p-2}u_nh - |v_n|^{p-2}v_nh) \rightarrow I_\alpha * (|u|^{p-2}uh) \end{cases} \text{ strongly in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \quad (3.2)$$

Also, by Lemma 3.3 we have

$$|u_n|^{p-2}u_nh \rightharpoonup |u|^{p-2}uh, \quad |v_n|^p \rightharpoonup 0, \quad |v_n|^{p-2}v_nh \rightharpoonup 0 \quad \text{weakly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \quad (3.3)$$

Combining (3.2)-(3.3) we find

$$\begin{cases} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)] (|u_n|^{p-2}u_nh - |v_n|^{p-2}v_nh) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2}uh, \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)] |v_n|^{p-2}v_nh = 0, \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^{p-2}u_nh - |v_n|^{p-2}v_nh)] |v_n|^p = 0. \end{cases} \quad (3.4)$$

By Hölder's inequality and Hardy-Littlewood-Sobolev inequality (2.6) with $s = t = \frac{2N}{N+\alpha}$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p) |v_n|^{p-2}v_nh \right| &\leq \|v_n\|_{\frac{2Np}{N+\alpha}}^p \| |v_n|^{p-1}h \|_{\frac{2N}{N+\alpha}} \\ &\leq C \| |v_n|^{p-1}h \|_{\frac{2N}{N+\alpha}}. \end{aligned} \quad (3.5)$$

On the other hand, by Lemma 3.3 we have $v_n^{\frac{2N(p-1)}{N+\alpha}} \rightharpoonup 0$ weakly in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$ so

$$\| |v_n|^{p-1}h \|_{\frac{2N}{N+\alpha}} = \left(\int_{\mathbb{R}^N} |v_n|^{\frac{2N(p-1)}{N+\alpha}} |h|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \rightarrow 0.$$

Thus, from (3.5) have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p) |v_n|^{p-2}v_nh = 0. \quad (3.6)$$

Passing to the limit in (3.1), from (3.4) and (3.6) we reach the conclusion. \square

4 Proof of Theorem 2.1

4.1 A Pohozaev identity

The main tool in proving Theorem 2.1 is the following Pohozaev type identity.

Proposition 4.1. *Let (u, ϕ) be a solution of (1.1) that satisfies (2.3)-(2.4). Then*

$$\int_{\mathbb{R}^N} \left(\frac{N-2}{2} |\nabla u|^2 + \frac{N}{2} |u|^2 \right) + \frac{N+2}{2q} \int_{\mathbb{R}^N} K(x) \phi |u|^q + \frac{1}{q} \int_{\mathbb{R}^N} \phi |u|^q x \cdot \nabla K(x) = \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p.$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}^N)$ be such that $\varphi \equiv 1$ on $B_1(0)$. For $\lambda > 0$ set

$$v_\lambda(x) = \varphi(\lambda x) x \cdot \nabla u.$$

Then $v_\lambda \in W^{1,2}(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ and from the first equation of (1.1) we have

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda + \int_{\mathbb{R}^N} u v_\lambda + \int_{\mathbb{R}^N} K(x) \phi |u|^{q-2} u v_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2} u v_\lambda. \quad (4.1)$$

Let us next analyse term by term the above equation. Since $u \in W_{loc}^{2,2}(\mathbb{R}^N)$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda &= \int_{\mathbb{R}^N} \varphi(\lambda x) \left[|\nabla u|^2 + x \cdot \nabla \left(\frac{|\nabla u|^2}{2} \right) (x) \right] dx \\ &= - \int_{\mathbb{R}^N} [\lambda x \cdot \nabla \varphi(\lambda x) + (N-2) \varphi(\lambda x)] \frac{|\nabla u(x)|^2}{2} dx. \end{aligned}$$

By Lebesgue dominated convergence theorem, we find

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda = - \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2.$$

Next,

$$\begin{aligned} \int_{\mathbb{R}^N} u v_\lambda &= \int_{\mathbb{R}^N} u(x) \varphi(\lambda x) x \cdot \nabla u(x) dx \\ &= \int_{\mathbb{R}^N} \varphi(\lambda x) x \cdot \nabla \left(\frac{|u|^2}{2} \right) (x) dx \\ &= - \int_{\mathbb{R}^N} [\lambda x \cdot \nabla \varphi(\lambda x) + N \varphi(\lambda x)] \frac{|u(x)|^2}{2} dx. \end{aligned}$$

Again by Lebesgue dominated convergence theorem we deduce

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} u v_\lambda = - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2.$$

Further we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2} u v_\lambda \\
&= \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y) |u|^p(y) \varphi(\lambda x) x \cdot \nabla(|u|^p)(x) dx dy \\
&= \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y) \left\{ |u|^p(y) \varphi(\lambda x) x \cdot \nabla(|u|^p)(x) + |u|^p(x) \varphi(\lambda y) y \cdot \nabla(|u|^p)(y) \right\} dx dy \\
&= -\frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y) |u|^p(x) |u|^p(y) \left[N \varphi(\lambda x) + \lambda x \cdot \nabla \varphi(\lambda x) \right] dx dy \\
&\quad + \frac{N-\alpha}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y)(x\varphi(\lambda x) - y\varphi(\lambda y))}{|x-y|^2} I_\alpha(x-y) |u|^p(x) |u|^p(y) dx dy,
\end{aligned}$$

which yields

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2} u v_\lambda = -\frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p.$$

Note that the regularity of K, u, ϕ and the second equation of (1.1) allow us to derive

$$\phi = I_2 * (K|u|^q).$$

Thus, we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} K(x) \phi |u|^{q-2} u v_\lambda \\
&= \frac{1}{q} \int_{\mathbb{R}^N} (I_2 * K|u|^q) K(x) \varphi(\lambda x) x \cdot \nabla(|u|^q) dx \\
&= \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_2(x-y) K(x) K(y) |u|^q(y) \varphi(\lambda x) x \cdot \nabla(|u|^q)(x) dx dy \\
&= \frac{1}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_2(x-y) K(x) K(y) \left\{ |u|^q(y) \varphi(\lambda x) x \cdot \nabla(|u|^q)(x) + |u|^q(x) \varphi(\lambda y) y \cdot \nabla(|u|^q)(y) \right\} dx dy \\
&= -\frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_2(x-y) K(y) |u|^q(x) |u|^q(y) \left[N K(x) \varphi(\lambda x) + K(x) \lambda x \cdot \nabla \varphi(\lambda x) + \varphi(\lambda x) x \cdot \nabla K(x) \right] dx dy \\
&\quad + \frac{N-2}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y)(x\varphi(\lambda x) - y\varphi(\lambda y))}{|x-y|^2} I_2(x-y) K(x) K(y) |u|^p(x) |u|^p(y) dx dy \\
&= -\frac{1}{q} \int_{\mathbb{R}^N} \phi(x) |u|^q(x) \left[N K(x) \varphi(\lambda x) + K(x) \lambda x \cdot \nabla \varphi(\lambda x) + \varphi(\lambda x) x \cdot \nabla K(x) \right] dx \\
&\quad + \frac{N-2}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y)(x\varphi(\lambda x) - y\varphi(\lambda y))}{|x-y|^2} I_2(x-y) K(x) K(y) |u|^p(x) |u|^p(y) dx dy.
\end{aligned}$$

We obtain

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} K(x) \phi |u|^{q-2} u v_\lambda = -\frac{N+2}{2q} \int_{\mathbb{R}^N} K(x) \phi |u|^q - \frac{1}{q} \int_{\mathbb{R}^N} \phi |u|^q x \cdot \nabla K(x).$$

Passing now to the limit in (4.1) we obtain the conclusion. \square

4.2 Proof of Theorem 2.1 completed

Let (u, ϕ) be a solution of (1.1) which satisfies (2.3)-(2.4). It is enough to show that $u \equiv 0$ as the second equation of (1.1) together with $\phi \in H^1(\mathbb{R}^N)$ will imply $\phi \equiv 0$. Suppose by contradiction that the solution (u, ϕ) satisfies $u \not\equiv 0$.

For convenience, let us denote

$$A(u) = \int_{\mathbb{R}^N} K(x) \phi |u|^q, \quad B(u) = \int_{\mathbb{R}^N} \phi |u|^q x \cdot \nabla K(x), \quad C(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p. \quad (4.2)$$

From Proposition 4.1 we have

$$\frac{N-2}{2} \|\nabla u\|_2^2 + \frac{N}{2} \|u\|_2^2 + \frac{N+2}{2q} A(u) + \frac{1}{q} B(u) = \frac{N+\alpha}{2p} C(u). \quad (4.3)$$

Since u is a solution of (2.10) we also have

$$C(u) = \|u\|^2 + A(u). \quad (4.4)$$

(i) Assume $x \cdot \nabla K(x) + \gamma K(x) \geq 0$ in \mathbb{R}^N for some $\gamma \in (-\infty, \frac{N+2}{2})$ and that (2.1) holds. Then

$$B(u) \geq -\gamma A(u)$$

so that from (4.3) and (4.4) we obtain

$$\frac{N+\alpha}{2p} (\|u\|^2 + A(u)) > \frac{N-2}{2} \|u\|^2 + \frac{N+2-2\gamma}{2q} A(u)$$

that is,

$$\frac{N+\alpha-p(N-2)}{p} \|u\|^2 > \left(\frac{N+2-2\gamma}{q} - \frac{N+\alpha}{p} \right) A(u).$$

But this last inequality is impossible since $\|u\| > 0$, $A(u) \geq 0$ and p, q, N, α, γ satisfy (2.1).

(ii) Assume $x \cdot \nabla K(x) + \gamma K(x) \leq 0$ in \mathbb{R}^N for some $\gamma \in \mathbb{R}$ and that (2.2) holds. It follows that

$$B(u) \leq -\gamma A(u)$$

so that (4.3) together with (4.4) yield

$$\frac{N+\alpha}{2p} (\|u\|^2 + A(u)) < \frac{N}{2} \|u\|^2 + \frac{N+2-2\gamma}{2q} A(u)$$

that is,

$$\frac{N+\alpha-pN}{p} \|u\|^2 < \left(\frac{N+2-2\gamma}{q} - \frac{N+\alpha}{p} \right) A(u).$$

Note that the above inequality is impossible since $\|u\| > 0$, $A(u) \geq 0$ and p, q, N, α, γ satisfy (2.2). This concludes our proof.

5 Proof of Theorem 2.2

5.1 The Nehari manifold associated with (2.10)

Define the Nehari manifold associated with \mathcal{J} as

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle = 0\} \quad (5.1)$$

and let

$$m_{\mathcal{J}} = \inf_{u \in \mathcal{N}} \mathcal{J}(u).$$

Remark that for $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$ we have

$$\langle \mathcal{J}'(tu), tu \rangle = t^2 \|u\|^2 + t^{2q} \int_{\mathbb{R}^N} K(x) \phi_u |u|^{q-2} u - t^{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p.$$

Since $p > q > 1$, the equation $\langle \mathcal{J}'(tu), tu \rangle = 0$ has a unique positive solution $t = t(u)$ and the corresponding element $t(u)u \in \mathcal{N}$ is called the *projection of u on \mathcal{N}* . The main properties of the Nehari manifold which we use in this paper are stated below.

Proposition 5.1. (i) $\mathcal{J}|_{\mathcal{N}}$ is bounded from below by a positive constant;

(ii) If u is a critical point of \mathcal{J} in \mathcal{N} then u is a free critical point of \mathcal{J} ;

Proof. (i) Using the Hardy-Littlewood-Sobolev inequality (2.6) together with the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, for any $u \in \mathcal{N}$ we have

$$\begin{aligned} 0 = \langle \mathcal{J}'(u), u \rangle &= \|u\|^2 + \int_{\mathbb{R}^N} K(x) \phi_u |u|^q - \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \\ &\geq \|u\|^2 - C \|u\|^{2p}. \end{aligned}$$

Hence, there exists $C_0 > 0$ such that

$$\|u\| \geq C_0 > 0 \quad \text{for all } u \in \mathcal{N}. \quad (5.2)$$

Using this fact we have

$$\begin{aligned} \mathcal{J}(u) &= \left(\frac{1}{2} - \frac{1}{2p}\right) \|u\|^2 + \left(\frac{1}{2q} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} K(x) \phi_u |u|^q \\ &\geq \left(\frac{1}{2} - \frac{1}{2p}\right) C_0^2 > 0. \end{aligned}$$

(ii) For $u \in H^1(\mathbb{R}^N)$ let $\mathcal{G}(u) = \langle \mathcal{J}'(u), u \rangle$. If $u \in \mathcal{N}$, by (5.2) we obtain

$$\begin{aligned} \langle \mathcal{G}'(u), u \rangle &= 2\|u\|^2 + 2q \int_{\mathbb{R}^N} K(x) \phi_u |u|^q - 2p \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \\ &= 2(1-q)\|u\|^2 - 2(p-q) \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \\ &\leq -2(q-1)\|u\|^2 \\ &< -2(q-1)C_0. \end{aligned} \quad (5.3)$$

Assume now that $u \in \mathcal{N}$ is a critical point of \mathcal{J} in \mathcal{N} . By the Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that $\mathcal{J}'(u) = \lambda \mathcal{G}'(u)$. In particular $\langle \mathcal{J}'(u), u \rangle = \lambda \langle \mathcal{G}'(u), u \rangle$. Since $\langle \mathcal{G}'(u), u \rangle < 0$, it follows that $\lambda = 0$ so $\mathcal{J}'(u) = 0$. \square

5.2 A compactness result

Let

$$\mathcal{E} : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}, \quad \mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p,$$

be the energy functional corresponding to (1.3). Also, consider its Nehari manifold

$$\mathcal{N}_\mathcal{E} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{E}'(u), u \rangle = 0\}$$

and let

$$m_\mathcal{E} = \inf_{u \in \mathcal{N}_\mathcal{E}} \mathcal{E}(u).$$

Proposition 5.2. *Let $(u_n) \subset \mathcal{N}$ be a (PS) sequence of $\mathcal{J}|_{\mathcal{N}}$, that is,*

- (a) $(\mathcal{J}(u_n))$ is bounded;
- (b) $(\mathcal{J}|_{\mathcal{N}})'(u_n) \rightarrow 0$ strongly in $H^{-1}(\mathbb{R}^N)$.

Then, there exists a solution $u \in H^1(\mathbb{R}^N)$ of (2.10) such that replacing (u_n) with a subsequence the following alternative holds

- (1) *either $u_n \rightarrow u$ strongly in $H^1(\mathbb{R}^N)$;*

or

- (2) *$u_n \rightharpoonup u$ weakly (but not strongly) in $H^1(\mathbb{R}^N)$ and there exists a positive integer $k \geq 1$, k functions $u_1, u_2, \dots, u_k \in H^1(\mathbb{R}^N)$ which are nontrivial weak solutions to (1.3) and k sequence of points $(y_{n,1}), (y_{n,2}), \dots, (y_{n,k}) \subset \mathbb{R}^N$ such that:*

- (i) $|y_{n,j}| \rightarrow \infty$ and $|y_{n,j} - y_{n,i}| \rightarrow \infty$ if $i \neq j$, $n \rightarrow \infty$;

- (ii) $u_n - \sum_{j=1}^k u_j(\cdot + y_{n,j}) \rightarrow u$ in $H^1(\mathbb{R}^N)$;

- (iii) $\mathcal{J}(u_n) \rightarrow \mathcal{J}(u) + \sum_{j=1}^k \mathcal{E}(u_j)$;

Proof. Because (u_n) is bounded in $H^1(\mathbb{R}^N)$, there exists $u \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H^1(\mathbb{R}^N), \\ u_n \rightharpoonup u & \text{weakly in } L^s(\mathbb{R}^N), \ 2 \leq s \leq 2^*, \\ u_n \rightarrow u & \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (5.4)$$

We also need the following result:

Lemma 5.1. *We have*

$$(i) \int_{\mathbb{R}^N} K(x) \phi_{u_n} |u_n|^q = \int_{\mathbb{R}^N} K(x) \phi_u |u|^q + o(1);$$

$$(ii) \int_{\mathbb{R}^N} K(x) \phi_{u_n} |u_n|^{q-2} u_n h = \int_{\mathbb{R}^N} K(x) \phi_u |u|^{q-2} u h + o(1), \text{ for all } h \in H^1(\mathbb{R}^N).$$

Proof. We shall prove only (ii) as the (i) part is similar.

Note first that

$$\left| \int_{\mathbb{R}^N} K(x) \phi_{u_n} |u_n|^{q-2} u_n h - \int_{\mathbb{R}^N} K(x) \phi_u |u|^{q-2} u h \right| \leq \int_{\mathbb{R}^N} |K(x)| |\phi_{u_n} - \phi_u| |u_n|^{q-1} |h|$$

$$+ \left| \int_{\mathbb{R}^N} K(x) \phi_u h (|u_n|^{q-2} u_n - |u|^{q-2} u) \right|. \quad (5.5)$$

Using Lemma 3.1(iii) and Hölder's inequality we find

$$\int_{\mathbb{R}^N} |K(x)| |\phi_{u_n} - \phi_u| |u_n|^{q-1} |h| \leq \|K\|_r \|\phi_{u_n} - \phi_u\|_{2^*} \|u_n|^{q-1}\|_{\frac{2^*}{q-1}} \|h\|_{2^*}$$

$$= \|K\|_r \|\phi_{u_n} - \phi_u\|_{2^*} \|u_n\|_{2^*} \|h\|_{2^*} = o(1). \quad (5.6)$$

By Lemma 3.3 we have $|u_n|^{q-2} u_n \rightharpoonup |u|^{q-2} u$ weakly in $L^{\frac{2^*}{q-1}}(\mathbb{R}^N)$.

Since $K(x) \phi_u h \in L^{\frac{2^*}{2^*-(q-1)}}(\mathbb{R}^N)$ it follows that

$$\int_{\mathbb{R}^N} K(x) \phi_u h (|u_n|^{q-2} u_n - |u|^{q-2} u) = o(1). \quad (5.7)$$

Now, the proof follows by combining (5.5)-(5.7). \square

We now return to the proof of Proposition 5.2. By (5.4), Lemma 3.6 and Lemma 5.1(ii) it follows that $\mathcal{J}'(u) = 0$ so $u \in H^1(\mathbb{R}^N)$ is a solution of (2.10).

If $u_n \rightarrow u$ strongly in $H^1(\mathbb{R}^N)$ then the first alternative in the statement of Proposition 5.2 holds and we are done. Assume in the following that (u_n) does not converge strongly in $H^1(\mathbb{R}^N)$ to u and define $z_{n,1} = u_n - u$. Then $(z_{n,1})$ converges weakly and not strongly to zero in $H^1(\mathbb{R}^N)$ and

$$\|u_n\|^2 = \|u\|^2 + \|z_{n,1}\|^2 + o(1). \quad (5.8)$$

By Lemma 3.5 we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p + \int_{\mathbb{R}^N} (I_\alpha * |z_{n,1}|^p) |z_{n,1}|^p + o(1). \quad (5.9)$$

Using (5.8), (5.9) and Lemma 5.1(i) we deduce

$$\mathcal{J}(u_n) = \mathcal{J}(u) + \mathcal{E}(z_{n,1}) + o(1). \quad (5.10)$$

For any $h \in H^1(\mathbb{R}^N)$, by Lemma 3.6 and Lemma 5.1(ii) we have

$$\langle \mathcal{E}'(z_{n,1}), h \rangle = o(1). \quad (5.11)$$

Next, by Lemma 3.5 and Lemma 5.1(i) we have

$$\begin{aligned} 0 &= \langle \mathcal{J}'(u_n), u_n \rangle = \langle \mathcal{J}'(u), u \rangle + \langle \mathcal{E}'(z_{n,1}), z_{n,1} \rangle + o(1) \\ &= \langle \mathcal{E}'(z_{n,1}), z_{n,1} \rangle + o(1), \end{aligned}$$

which yields

$$\langle \mathcal{E}'(z_{n,1}), z_{n,1} \rangle = o(1). \quad (5.12)$$

Let

$$\delta := \limsup_{n \rightarrow \infty} \left(\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |z_{n,1}|^{\frac{2Np}{N+\alpha}} \right) \geq 0.$$

We claim that $\delta > 0$. Indeed, if $\delta = 0$, by Lemma 3.2 we deduce $z_{n,1} \rightarrow 0$ strongly in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Then, by Hardy-Littlewood-Sobolev inequality (2.6) we find

$$\int_{\mathbb{R}^N} (I_\alpha * |z_{n,1}|^p) |z_{n,1}|^p = o(1).$$

This fact combined with (5.12) yields $z_{n,1} \rightarrow 0$ strongly in $H^1(\mathbb{R}^N)$ in contradiction to our assumption.

Hence, $\delta > 0$ so that we may find $y_{n,1} \in \mathbb{R}^N$ with

$$\int_{B_1(y_{n,1})} |z_{n,1}|^{\frac{2Np}{N+\alpha}} > \frac{\delta}{2}. \quad (5.13)$$

Considering the sequence $(z_{n,1}(\cdot + y_{n,1}))$, there exists $u_1 \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$\begin{aligned} z_{n,1}(\cdot + y_{n,1}) &\rightharpoonup u_1 && \text{weakly in } H^1(\mathbb{R}^N), \\ z_{n,1}(\cdot + y_{n,1}) &\rightarrow u_1 && \text{strongly in } L^{\frac{2Np}{N+\alpha}}_{loc}(\mathbb{R}^N), \\ z_{n,1}(\cdot + y_{n,1}) &\rightarrow u_1 && \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Passing to the limit in (5.13) we find

$$\int_{B_1(0)} |u_1|^{\frac{2Np}{N+\alpha}} \geq \frac{\delta}{2},$$

so $u_1 \not\equiv 0$. Also, since $(z_{n,1})$ converges weakly to zero in $H^1(\mathbb{R}^N)$ it follows that $(y_{n,1})$ is unbounded. Passing to a subsequence we may assume $|y_{n,1}| \rightarrow \infty$. From (5.12) we also obtain $\mathcal{E}'(u_1) = 0$, so u_1 is a nontrivial solution of (1.3).

Set next

$$z_{n,2}(x) = z_{n,1}(x) - u_1(x - y_{n,1}).$$

As above we have

$$\|z_{n,1}\|^2 = \|u_1\|^2 + \|z_{n,2}\|^2 + o(1).$$

and by Lemma 3.5 we derive

$$\int_{\mathbb{R}^N} (I_\alpha * |z_{n,1}|^p) |z_{n,1}|^p = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p + \int_{\mathbb{R}^N} (I_\alpha * |z_{n,2}|^p) |z_{n,2}|^p + o(1).$$

Thus,

$$\mathcal{E}(z_{n,1}) = \mathcal{E}(u_1) + \mathcal{E}(z_{n,2}) + o(1)$$

so, by (5.10) one has

$$\mathcal{J}(u_n) = \mathcal{J}(u) + \mathcal{E}(u_1) + \mathcal{E}(z_{n,2}) + o(1).$$

Using the above techniques one can also derive

$$\langle \mathcal{E}'(z_{n,2}), h \rangle = o(1) \quad \text{for any } h \in H^1(\mathbb{R}^N)$$

and

$$\langle \mathcal{E}'(z_{n,2}), z_{n,2} \rangle = o(1).$$

If $(z_{n,2})$ converges strongly to zero, the proof finishes (and take $k = 1$ in the statement of Proposition 5.2). Assuming that $z_{n,2} \rightharpoonup 0$ weakly and not strongly in $H^1(\mathbb{R}^N)$, we iterate the process. In k number of steps we find a set of sequences $(y_{n,j}) \subset \mathbb{R}^N$, $1 \leq j \leq k$ with

$$|y_{n,j}| \rightarrow \infty \quad \text{and} \quad |y_{n,i} - y_{n,j}| \rightarrow \infty \quad \text{as } i \neq j, n \rightarrow \infty$$

and k nontrivial solutions $u_1, u_2, \dots, u_k \in H^1(\mathbb{R}^N)$ of (1.3) such that, denoting

$$z_{n,j}(x) := z_{n,j-1}(x) - u_{j-1}(x - y_{n,j-1}), \quad 2 \leq j \leq k,$$

we have

$$z_{n,j}(x + y_{n,j}) \rightharpoonup u_j \quad \text{weakly in } H^1(\mathbb{R}^N)$$

and

$$\mathcal{J}(u_n) = \mathcal{J}(u) + \sum_{j=1}^k \mathcal{E}(u_j) + \mathcal{E}(z_{n,k}) + o(1).$$

Since $\mathcal{E}(u_j) \geq m_{\mathcal{E}}$ and $(\mathcal{J}(u_n))$ is bounded, the process can be iterated only a finite number of times. This concludes our proof. \square

Corollary 3. *Let $c \in (0, m_{\mathcal{E}})$. Then, any $(PS)_c$ sequence of $\mathcal{J}|_{\mathcal{N}}$ is relatively compact.*

Proof. Let (u_n) be a $(PS)_c$ sequence of $\mathcal{J}|_{\mathcal{N}}$. Since $\mathcal{E}(u_j) \geq m_{\mathcal{E}}$ in Proposition 5.2, it follows that up to a subsequence $u_n \rightarrow u$ strongly in $H^1(\mathbb{R}^N)$ and u is a solution of (2.10). \square

5.3 Proof of Theorem 2.2 completed

The proof of Theorem 2.2 relies essentially on the following result.

Lemma 5.2. *There exists $M > 0$ such that if $K \in L^r(\mathbb{R}^N)$ and $\|K\|_r < M$ then*

$$m_{\mathcal{J}} < m_{\mathcal{E}}.$$

Proof. Denote by $w \in H^1(\mathbb{R}^N)$ the ground state solution of (1.3). By [23, Theorem 1] we know that such a ground state exists. Let tw be the projection of w on \mathcal{N} , that is, $t = t(w) > 0$ is the unique real number such that $tw \in \mathcal{N}$ (with \mathcal{N} defined in (5.1)). Denote

$$A(w) = \int_{\mathbb{R}^N} K(x) \phi_w |w|^q, \quad B(w) = \int_{\mathbb{R}^N} (I_\alpha * |w|^p) |w|^p.$$

Since $w \in \mathcal{N}_{\mathcal{E}}$ and $tw \in \mathcal{N}$ we have

$$\|w\|^2 = B(w) \tag{5.14}$$

and

$$t^2 \|w\|^2 + t^{2q} A(w) = t^{2p} B(w).$$

From the above equalities we find $t > 1$. Further, by Hölder inequality we have

$$A(w) \leq \|K\|_r \|\phi_w\|_{2^*} \|w\|_{2^*}^q. \tag{5.15}$$

From (5.14) and (5.15) we deduce

$$\begin{aligned} m_{\mathcal{J}} &\leq \mathcal{J}(tw) = \frac{1}{2} t^2 \|w\|^2 + \frac{1}{2q} t^{2q} A(w) - \frac{1}{2p} t^{2p} B(w) \\ &= \left(\frac{t^2}{2} - \frac{t^{2p}}{2p} \right) \|w\|^2 + \frac{t^{2q}}{2q} \|K\|_r \|\phi_w\|_{2^*} \|w\|_{2^*}^q. \end{aligned}$$

Since $t > 1$, by letting $\|K\|_r$ small, it follows that

$$m_{\mathcal{J}} < \left(\frac{1}{2} - \frac{1}{2p} \right) \|w\|^2 = \mathcal{E}(w) = m_{\mathcal{E}}.$$

□

By Ekeland Variational Principle, for any $n \geq 1$ there exists $u_n \in \mathcal{N}$ such that

$$\begin{aligned} \mathcal{J}(u_n) &\leq m_{\mathcal{J}} + \frac{1}{n} && \text{for all } n \geq 1, \\ \mathcal{J}(u_n) &\leq \mathcal{J}(v) + \frac{1}{n} \|v - u_n\| && \text{for all } v \in \mathcal{N}, n \geq 1. \end{aligned}$$

From here we easily deduce that $(u_n) \subset \mathcal{N}$ is a $(PS)_{m_{\mathcal{J}}}$ sequence for $\mathcal{J}|_{\mathcal{N}}$. Using Lemma 5.2 and Corollary 3 it follows that up to a subsequence, (u_n) converges strongly to some $u \in H^1(\mathbb{R}^N)$ which is a ground state solution of \mathcal{J} .

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